Topics from harmonic analysis related to generalized Poincaré-Sobolev inequalities: Lecture III

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$$\frac{1}{2^n}\int_{x\in Q: f(x)>t} f(x)dx \leq t|\Omega_t| \leq 2\int_{x\in Q: f(x)>t/2} f(x)dx.$$

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Theorem Let $f \in BMO$, then for a dimensional constant c_n such that for each cube Q,

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$$H : L^\infty \to BMO$$

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- Observation: The L^p norm can be replaced by the weak norm $L^{p,\infty}$.

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Another possibility is to use the Taylor expansion of e^{cf}...

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where the functions g_Q and b_Q are defined as usual. We have that

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Note that this definition makes sense since the cubes $\{Q_j\}$ are disjoint, so any $x \in \Omega_L$ belongs to only one Q_j. Also note that condition on the right hand side implies that

$$g_Q(x) \leq 2^n L \tag{3}$$

for almost all $x \in Q$. The function b_Q is determined by this choice of g_Q as the difference

$$b_Q = f - f_Q - g_Q,$$

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END THIRD LECTURE